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TOWARDS A MODULI THEORETIC CHARACTERIZATION OF A RATIONAL PRIME $\mathbb{Q}$-FANO 3-FOLD OF GENUS SIX WITH ONE $\frac{1}{2}(1,1,1)$-SINGULARITY

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1. DEFINITIONS, MOTIVATIONS,...

Definition 1.1. A projective 3-fold X is called a $\mathbb{Q}$-*Fano* 3-*fold*, or simply, *Fano* 3-*fold* if X has only terminal singularities and the anti-canonical divisor $-K_X$ is ample.

The *genus* $g(X)$ of a Fano 3-fold X is defined to be $h^0(-K_X)-2$. Note that, in case X is smooth, $2g(X)-2=(-K_X)^3$ by the Riemann-Roch theorem and the Kodaira vanishing theorem.

A Fano 3-fold is called *prime* if the group of numerical equivalence classes of $\mathbb{Q}$-Cartier Weil divisors is generated by the anti-canonical class. A quartic hypersurface in $\mathbb{P}^4$ is a simple but interesting example of a prime Fano 3-fold.

Aim 1.2. Find as many as possible (prime) Fano 3-folds X which can be recovered from data on curves C 'characteristic' for X, ideally, as a moduli space of some objects on C.

S. Mukai found beautiful examples of smooth prime Fano 3-folds for which such characterizaztions are possible. I will explain his discovery. Before that, let me show how to find candidates of C to recover X. It is done by variants of the Fano-Iskovskih double projection from a line on a prime Fano 3-fold (Fano, Iskovskih, Takeuchi).

Example 1.3. See [IP99] or [Take89] for details. Let X be a smooth prime Fano 3-fold with very ample $-K_X$ in this example. I embed X in $\mathbb{P}^{g(X)+1}$ by the anti-canonical linear system.

(1) ($g(X)=7$) Let q be a general conic (with respect to the anti-canonical embedding) on X and consider the rational map defined by the linear system $|-2K_X-3q|$, the sublinear system of $|-2K_X|$ consisting of the members with multiplicities 3 along q. The image turns out to be Q^3, a smooth quadric 3-fold. This rational map is

the composite of the three elementary rational maps as follows:

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X \arrow[rrr, dashed, "\Phi_{|-2K_X-3q|}"'] & & & Q^3, \end{tikzcd}$$

where f is the blow-up along q, $Y \dashrightarrow Y'$ a flop, and f' is the blow-up of Q^3 along a smooth curve C_q of genus 7 and degree 10. For this example, I will take C_q as C.

(2) ($g(X) = 9$) Let l be a general line and consider the rational map defined by the linear system $|-K_X - 2l|$.[1] Similarly to the above case, I obtain the following diagram:

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X \arrow[rrr, dashed, "\Phi_{|-K_X-2l|}"'] & & & \mathbb{P}^3, \end{tikzcd}$$

where f is the blow-up along l, $Y \dashrightarrow Y'$ a flop, and f' is the blow-up of $\mathbb{P}^3$ along a smooth curve C_l of genus 3 and degree 7. For this example, I will take C_l as C.

(3) ($g(X) = 12$) In this example, I consider also the rational map defined by the linear system $|-K_X - 2l|$ for a general line l and I obtain the following diagram:

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X \arrow[rrr, dashed, "\Phi_{|-K_X-2l|}"'] & & & B_5, \end{tikzcd}$$

where f is the blow-up along l, $Y \dashrightarrow Y'$ a flop, B_5 is a smooth quintic del Pezzo 3-fold, [2] and f' is the blow-up of B_5 along a smooth curve C_l of genus 0 and degree 5.[3] For this example, I will **not** take C_l as C. Instead I will take the Hilbert scheme of lines on X, which I can compute by the diagram noting general lines are transformed to general lines on B_5 intersecting C. Consequently, C is a plane quartic curve and is smooth if X is general in the moduli.

[1] This rational map is the so-called double projection from a line.

[2] A quintic del Pezzo 3-fold is a Fano 3-fold such that $-K_X = 2H$, where $H \in \operatorname{Pic} X$ and $H^3 = 5$.

[3] The degree is with respect to H.

Keep in mind the notation of Example 1.3. Mukai's theorem (with comments) is the following (see [Muk01]):

Theorem 1.4. (1) $(g(X) = 7)$

$$X \simeq \{[\mathcal{E}] \mid \mathcal{E} \textit{ is a rank } 2 \textit{ semi-stable vector bundle on } C \\ \textit{with } \det \mathcal{E} = K_C \textit{ and } h^0(\mathcal{E}) \geq 5\}.$$

This is an example of non-abelian Brill-Noether locus.

(2) $(g(X) = 9)$ In this case, X cannot be recovered from C because the moduli number of X is 12 [4] and the moduli number of C is 6. Thus some data on C is needed. Mukai showed the following and used it as a data to recover X:
The Hilbert scheme of conics on X is the smooth surface $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a Nagata stable vector bundle of rank 2.[5]

$$X \simeq \{[\mathcal{E}] \mid \mathcal{E} \textit{ is a rank } 2 \textit{ semi-stable vector bundle on } C \\ \textit{with } \det \mathcal{E} = \det \mathcal{F} + K_C \textit{ and } \dim Hom(\mathcal{F}, \mathcal{E}) \geq 3\}.$$

This is an example of another kind of non-abelian Brill-Noether locus.

(3) $(g(X) = 12)$ Though the moduli numbers of X and C are the same, X cannot be recovered from C. As a data to recover X, Mukai obtained the following:
There exists a unique theta-characteristic θ on C with $h^0(\theta) = 0$ such that inside $C \times C$, [6]

$$\{([l_1], [l_2]) \mid l_1 \neq l_2, l_1 \cap l_2 \neq \emptyset\} = \{([l_1], [l_2]) \mid h^0(\theta + [l_1] - [l_2]) > 0\}.$$

A classic result of Scorza asserts that there exists a unique quartic curve Γ living in the same $\mathbb{P}^2$ as C associated to the pair (C, θ) (see [DK93]). Let F be a defining equation of Γ.
X is isomorphic to the closure in $\mathrm{Hilb}^6 \check{\mathbb{P}}^2$ of the following:

$$\{\langle \check{l_1}, \dots \check{l_6} \rangle \mid l_1^4 + \dots + l_6^4 = F\},$$

where l_i is a linear form on $\mathbb{P}^2$ and $\check{l_i}$ is the point of $\check{\mathbb{P}}^2$ corresponding to l_i.

Remark. Mukai conjectured that there is a similar characterization of a prime Fano 3-fold X of genus 10 (see [Mukb]). In this case, C is a smooth curve of genus 2, which is the center of the blow-up of Q^3

[4] This can be computed by the diagram in Example 1.3.

[5] A vector bundle $\mathcal{F}$ of rank 2 is called *Nagata stable* if $\sigma^2 \geq 3 = g(C)$ for any section σ of $\mathbb{P}(\mathcal{F})$.

[6] Recall that C is the Hilbert scheme of lines on X.

appearing in the decomposition of the double projection from a general line.

2. Singular Fano 3-folds

I am tempted to find more examples of Fano 3-folds with characterzations as in Theorem 1.4 and more curves which are characteristic for some Fano 3-folds. In my thesis [Taka02a] and [Taka02b], I classified prime Fano 3-folds X with $g(X) \geq 2$ and with only $\frac{1}{2}$-singularities. [7] More precisely, I classified the type of the following diagram, which is a variant of the double projection from a line as in the previous section.

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X & & & X', \end{tikzcd}$$

where f is the blow-up at a $\frac{1}{2}$-singularity, $Y \dashrightarrow Y'$ is a flop or a composite of a flop and a flip, f' is a non-small extremal contraction. I present two examples. I denote by N the number of $\frac{1}{2}$-singularities.

Example 2.1. (1) ($g(X) = 8, N = 2$). The diagram is as follows:

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X & & & B_5, \end{tikzcd}$$

where $Y \dashrightarrow Y'$ is a composite of a flop and a flip, and f' is the blow-up along $C \simeq \mathbb{P}^1$ with $\deg C = 6$.

This diagram is very similar to the smooth prime Fano 3-fold of genus 12 (Example 1.3 (3)). Actually there are more similarities. I studied this case more in detail with Francesco Zucconi in Udine. I briefly explain our results. Assume that X is general in the moduli.

- The Hilbert scheme of 'lines'[8] is isomorphic to a smooth complete intersection of a smooth quadric and a cubic in $\mathbb{P}^3$. I choose as C this curve of genus 4.

[7] A $\frac{1}{2}$-singularity is, by definition, analytically isomorphic to the origin of $\mathbb{C}^3/(x,y,z) \sim (-x,-y,-z)$, where (x,y,z) is the coordinate of $\mathbb{C}^3$. Usually this is called a $\frac{1}{2}(1,1,1)$-singularity.

[8] Here, by a line, I mean a curve with degree 1 with respect to $-K_X$ and with arithmetic genus 0. There is a degenerate line, which is the union of two $\mathbb{P}^1$'s with degree $\frac{1}{2}$ with respect to $-K_X$.

genus six

- As in the case of the smooth prime Fano 3-fold of genus 12, there exists a unique theta-characteristic θ on C with $h^0(\theta) = 0$ such that inside $C \times C$,

$$\{([l_1],[l_2]) \mid l_1 \neq l_2, l_1 \cap l_2 \neq \emptyset\} = \{([l_1],[l_2]) \mid h^0(\theta + [l_1] - [l_2]) > 0\}.$$

  A classic result of Scorza and complementary works by Dolgachev and Kanev assert that there exists a unique quartic surface Γ living in the same $\mathbb{P}^3$ as C associated to the pair (C, θ). Let F be a defining equation of Γ.

- The Hilbert scheme S of 'conics' is the smooth surface obtained by blowing up $\mathbb{P}^2$ at 6 points lying on a smooth conic, and S is a weak del Pezzo surface of degree 3. Denote by $\overline{S}$ the anti-canonical model of S.
- As a characterization of X, we conjecture the following:

  Conjecture 2.2. 'An explicit birational model' of X can be embedded in $\mathrm{Hilb}^{10} S$ as the closure of the locus

$$\{\langle \check{l}_1, \dots \check{l}_{10} \rangle \mid l_1^4 + \cdots + l_{10}^4 = F, \check{l}_i \in S\},$$

  where l_i is a linear form on $\mathbb{P}^3$ and $\check{l}_i$ is the point of $\check{\mathbb{P}}^3$ corresponding to l_i.

(2) ($g(X) = 6, N = 1$) There are two type of Fano 3-folds with these invariants, one of which is birational to a smooth cubic 3-fold, another is rational. I only describe the latter case. The diagram is as follows:

$$\begin{tikzcd}
 & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\
X & & & Q^3,
\end{tikzcd}$$

where $Y \dashrightarrow Y'$ is a flop, and f' is the blow-up along a smooth curve C with $g(X) = 6$ and $\deg C = 9$. I will choose as a characteristic curve for X this C and I will go back to this case in the next section.

By looking at the list of Fano 3-folds with $g(X) \geq 2$ and with only $\frac{1}{2}$-singularities, I obtain the following range of genus of curves as the genus of characteric curves:

$$g(C) = 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Thus I hope that Fano 3-folds are useful for the study of curves with small genus.

3. Rational Fano 3-fold with genus 6 and with one $\frac{1}{2}$-singularity

From now on, let X be

$$\text{a rational Fano 3-fold of genus 6 and with one } \frac{1}{2}\text{-singularity.}$$

Assume that X is general in the moduli.

First I describe the diagram in Example 2.1 (2) more in detail. It is easy to show the following:

- The composite of the embedding $C \hookrightarrow Q^3 \hookrightarrow \mathbb{P}^4$ is defined by the linear system $|K_C - p|$, where p is a point of C.
- There exists a pencil of quadrics in $\mathbb{P}^4$ containing C. The interection of the quadrics in the pencil is a smooth del Pezzo surface S of degree 4. S is the strict transform of the f-exceptional divisor.
- There exist 5 tri-secants lines of C, which are contained in S. These are the images of flopping curves for $Y' \dashrightarrow Y$.
- C is isomorphic to a complete intersection in $G(5,2)$ defined by 4 hyperplanes and 1 quadric hypersurface. By [Muk93], this is equivalent to that C has no g^1_4, g^2_5 and C is not bi-elliptic.

The following is the main result of this article with comments:

Proposition 3.1. (A) In this case, X cannot be recovered from C because the moduli number of X is 17 [9] and the moduli number of C is 15. Thus some data on C is needed as in the case of the smooth prime Fano 3-fold of genus nine.

$(A1)$ *The Hilbert scheme $\mathcal{H}_{5/2}$ of $\frac{5}{2}$-curves* [10] *on X is the smooth surface $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a stable and globally generated vector bundle of rank 2 obtained as follows: let $\mathcal{F}_0$ be the restriction of the universal quotient bundle on $G(5,2)$ (now I consider that C is embedded in $G(5,2)$). $\mathcal{F}$ fits into the exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to k(p) \to 0.$$

$(A2)$ *X can be recovered from $\mathcal{F}$.*

Unfortunately in (A2), I did not succeed in recovering X as a moduli.

(B) As for the recovery as a moduli of X, I have the following weaker result than expected.

[9] This can be computed by the diagram in Example 2.1 (2).

[10] by a $\frac{5}{2}$-curve, I mean a curve with degree $\frac{5}{2}$ with respect to $-K_X$ and with arithmetic genus 0.

Let $g\colon \widetilde{\mathbb{P}^4} \to \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ along C, and $h\colon \widetilde{\mathbb{P}^4} \to Z$ the anti-flipping contraction of the strict transforms of 5 tri-secant lines of C. Let

$$M := \{[\mathcal{E}] \mid \mathcal{E} \textit{ is a rank 3 semi-stable vector bundle on } C \\ \textit{with } \det \mathcal{E} = K_C - p \textit{ and } h^0(\mathcal{E}) \geq 4\}.$$

There exists a finite birational morphism $Z \to M$.
Once I can prove M is normal, I have $Z \simeq M$. Since the anti-canonical model $\overline{Y}$ of Y is contained in Z, I belive that $\overline{Y}$ can be characterized as a moduli by using $\mathcal{F}$.

4. Outline of the proof of Proposition 3.1

For (A), it suffices to prove the following:
Let C be a general smooth curve of genus 6. In particular, C has no g^1_4 and g^2_5 and C is not bi-elliptic. Let p be a general point of C. Finally let $\mathcal{F}$ be a stable and globally generated bundle of rank 2 on C obtained as in the statement of Proposition 3.1 (A). Then there is an embedding $C \hookrightarrow Q^3$ such that by blowing up Q^3 along C, Q^3 can be biratinally transformed to a Fano 3-fold of genus 6 as in the diagram in Example 2.1 (2).

I only show the following diagram, from which the assertion is easily verified:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\Phi_{|K_C-p|}"'] & G(H^0(\mathcal{F}), 2) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}^4 \arrow[r] & \mathbb{P}^5, \end{tikzcd}$$

where $C \to G(H^0(\mathcal{F}), 2) \simeq G(4, 2)$ is defined by

$$x \mapsto C \to (H^0(\mathcal{F}) \to \mathcal{F}_x) \in G(H^0(\mathcal{F}), 2).$$

I define $Q^3 := G(H^0(\mathcal{F}), 2) \cap \mathbb{P}^4$.

I will explain why $\mathbb{P}(\mathcal{F}) \simeq \mathcal{H}_{5/2}$. By the diagram in Example 2.1 (2), I can show that a general $\frac{5}{2}$-curve on X is a birational transform of a general line on Q^3 intersecting C. Thus I explain how to attach to a point $s \in \mathbb{P}(\mathcal{F})$ a line l_s on Q^3 intersecting C. For a point $s \in \mathbb{P}(\mathcal{F})$, set

$$V_s := \{\sigma \in H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \mid s \in (\sigma)_0\} \subset H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \simeq H^0(\mathcal{F}).$$

Note that $\dim V_s = 3$ since $\mathcal{F}$ is globally generated. Set

$$l_s := G(2, V_s) \cap Q^3 \subset G(2, H^0(\mathcal{F})) = G(H^0(\mathcal{F}), 2),$$

which is a line since $G(2, V_s) \simeq \mathbb{P}^2$ and Q^3 does not contain a plane. Let $u := \pi(s)$, where $\pi \colon \mathbb{P}(\mathcal{F}) \to C$ is the natural projection. Note that $u = \ker(H^0(\mathcal{F}) \to \mathcal{F}_u)$ in $G(2, H^0(\mathcal{F}))$. Thus $u \in l_s \cap C$ since

$$\ker(H^0(\mathcal{F}) \to \mathcal{F}_u) = \{\sigma \in H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \mid \pi^{-1}(u) \subset (\sigma)_0\} \subset V_s.$$

Now I explain the proof of Proposition 3.1 (B), which depends on the two propositions.

I start from the preparation for the first proposition. Let U_t be the 4-dimensional subspace of $H^0(K-p)$ corresponding to $t \in \mathbb{P}^4 = \mathbb{P}^* H^0(K-p)$. Define $\mathcal{E}_t^\vee$ by

$$0 \to \mathcal{E}_t^\vee \to U_t \otimes \mathcal{O}_C \to K - p.$$

If $t \notin C$, then $U_t \otimes \mathcal{O}_C \to K-p$ is surjective, thus $\det \mathcal{E}_t = K_C - p$. If $t \in C$, then $\operatorname{Im}(U_t \otimes \mathcal{O}_C \to K-p) = K-p-t$, thus $\det \mathcal{E}_t = K_C - p - t$. Actually, Mukai constructs in [Muka] the vector bundle $\widetilde{\mathcal{E}}$ on $\widetilde{\mathbb{P}}^4 \times C$ such that for $t' \in \widetilde{\mathbb{P}}^4$, if $t := g(t') \notin C$, then $\widetilde{\mathcal{E}}_{t'} \simeq \mathcal{E}_t$, or if $t \in C$, then $\widetilde{\mathcal{E}}_{t'}$ fits into the exact sequence

$$0 \to \mathcal{E}_t \to \widetilde{\mathcal{E}}_{t'} \to k(t) \to 0.$$

Thus $\det \widetilde{\mathcal{E}}_{t'} = K_C - p$ for any $t' \in \widetilde{\mathbb{P}}^4$.

Proposition 4.1. *$\widetilde{\mathcal{E}}_{t'}$ is semi-stable for any $t' \in \widetilde{\mathbb{P}}^4$, and $\widetilde{\mathcal{E}}_{t'}$ is strictly semi-stable if and only if one of the following equivalent condition hold:*

(1) *there exists an exact sequence as follows:*

$$0 \to \delta - p \to \widetilde{\mathcal{E}}_{t'} \to \mathcal{G} \to 0,$$

where δ is a g^1_4 and $\mathcal{G}$ is a stable vector bundle of rank 2 uniquely determined by

$$0 \to \mathcal{G}^\vee \to H^0(K-\delta) \otimes \mathcal{O}_C \to K - \delta \to 0.$$

(2) *t' is on the strict transform of a tri-secant line of C.*

The correspondence between a g^1_4 in (1) and a tri-secant line in (2) is given as follows: for δ in (1), the unique member $|\delta - p|$ lies on a tri-secant line, and vice versa.

In particular, the S-equivalent classes of $\widetilde{\mathcal{E}}_{t'}$ is constant on the strict transform of a tri-secant line.

Proposition 4.2. *Let $\mathcal{E} \in M_C(3, K-p, 1)$,*

$$ev_{\mathcal{E}} := H^0(\mathcal{E}) \otimes \mathcal{O}_C \to \mathcal{E} \text{ and } \mathcal{E}_1 := \operatorname{Im} ev_{\mathcal{E}}.$$

Then $\dim H^0(C, \mathcal{E}) = 4$ and $\operatorname{rk} \mathcal{E}_1 = 3$. Moreover one of the following holds:

(1) *$ev_{\mathcal{E}}$ is surjective. In this case, $\mathcal{E}$ defines a point of $\mathbb{P}^4 \setminus C$.*

(2) $\operatorname{rk}\mathcal{E}_1 = 3$, $h^0(\mathcal{E}_1^\vee) = 0$ *and there exists an exact sequence as follows:*

$$0 \to \mathcal{E}_1^\vee \to \mathcal{O}_C^{\oplus 4} \to K_C - p - x \to 0$$

for a point $x \in C$.

(3) $\operatorname{rk}\mathcal{E}_1 = 3$ *and* $h^0(\mathcal{E}_1^\vee) > 0$, *and there exists an exact sequence as follows:*

$$0 \to \mathcal{G} \to \mathcal{E} \to \delta - p \to 0,$$

where δ *is a* g_4^1 *and* $\mathcal{G}$ *is a stable vector bundle of rank* 2 *uniquely determined by*

$$0 \to \mathcal{G}^\vee \to H^0(K - \delta) \otimes \mathcal{O}_C \to K - \delta \to 0.$$

I omit the proof of these propositions. I just mention that the proof are based on the so-called Castelnuovo's trick of the following type.

Lemma 4.3. *Let* $\mathcal{E}$ *be a rank* 2 *vector bundle on a smooth curve. Set* $r := h^0(\mathcal{E})$ *and* $s := \dim \operatorname{Im}(\wedge^2 H^0(\mathcal{E}) \to H^0(\wedge^2 \mathcal{E}))$. *If* $\dim G(2,r) = 2(r-2) \geq s$, *then there exists a* 2*-dimensional subaspace* V *of* $H^0(\mathcal{E})$ *such that* $\operatorname{Im}(V \otimes \mathcal{O}_C \to \mathcal{E})$ *is invertible.*

reference??

I continue the outline of the proof of Proposition 3.1 (B). The vector bundles in the cases (1) and (2) of Proposition 4.2 appear as $\widetilde{\mathcal{E}}_{t'}$ for some t'. The vector bundles in the case (3) are new but S-equivalent to strictly semi-stable $\widetilde{\mathcal{E}}_{t'}$ in Proposition 4.1. Hence we have the surjective morphism $\iota\colon \widetilde{\mathbb{P}}^4 \to M$. The fact $h^0(\mathcal{E}) = 4$ for $[\mathcal{E}] \in M$ (Proposition 4.2) implies that $\mathcal{E}_{t_1} \not\simeq \mathcal{E}_{t_2}$ for two points t_1, t_2 on $\mathbb{P}^4 \setminus C$ since U_{t_i} can be recovered by $\mathcal{E}_{t_i}$ as $U_{t_i} = H^0(\mathcal{E}_{t_i})^\vee$. Thus ι is birational. Moreover strictly semi-stable bundle in M are parameterized by the points on the strict transforms of tri-secants and their S-equivalence classes are constant on each strict transform, ι descends on Z. Since $\rho(Z) = 1$, the morphism ι is finite.

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